CONTINUITY OF A FUNCTION :→ Let, f be a real-valued bounded function defined on $[a,b]$. Let, c be any interior point of $[a,b]$. The function $f(x)$ is said to be continuous at $x=c$ if for any $\varepsilon>0$ $\exists$ a real number $\delta>0$ s.t.

$$|f(x)-f(c)|<\varepsilon \text{ whenever } |x-c|<\delta$$

and we write $\lim_{x\to c} f(x) = f(c)$.

Alternatively, the function $f(x)$ is said to be continuous at $x=c$ if $\lim_{x\to c+} f(x) = \lim_{x\to c-} f(x) = f(c)$.

Example :→ 1. Examine the continuity of the function

$$f(x) = \begin{cases} x+2, & \text{if } x\le 1 \\ x-2, & \text{if } 1<x<2 \\ 0, & \text{if } x\geq 2 \end{cases}$$

at $x=1$ and $x=2$.

Solution :→ For $x=1$,

$$\lim_{x\to 1-} f(x) = \lim_{x\to 1-} (x+2) = 1+2 = 3.$$

$$\lim_{x\to 1+} f(x) = \lim_{x\to 1+} (x-2) = 1-2 = -1.$$

Since $\lim_{x\to 1-} f(x) \neq \lim_{x\to 1+} f(x)$, so the function $f(x)$ is not continuous at $x=1$.

For $x=2$,

$$\lim_{x\to 2-} f(x) = \lim_{x\to 2-} (x-2) = 2-2 = 0.$$

and $\lim_{x\to 2+} f(x) = \lim_{x\to 2+} 0 = 0$.

Now $f(2)=0$.

Since $\lim_{x\to 2-} f(x) = \lim_{x\to 2+} f(x) = f(2)$, so the function $f(x)$ is continuous at $x=2$.

Discontinuity :→ The function $f(x)$ is said to be discontinuous at the point $x=c$, if it is not continuous at $x=c$.

Types of Discontinuity: There are three types of discontinuity. Those are

i) **Discontinuity of First Kind:** In this case both the limits $\lim_{x\to c+} f(x)$ and $\lim_{x\to c-} f(x)$ exist but they are unequal.

i.e. $\lim_{x\to c+} f(x) \neq \lim_{x\to c-} f(x)$.

The value $\left[\lim_{x\to c+} f(x) - \lim_{x\to c-} f(x)\right]$ is called the jump of discontinuity.

Example: A function $f(x)$ is defined as follows:

$$f(x) = \begin{cases} \frac{1}{2} - x, & \text{when } 0 < x < \frac{1}{2} \\ \frac{1}{2}, & \text{when } x = \frac{1}{2} \\ \frac{3}{2} - x, & \text{when } \frac{1}{2} < x < 1. \end{cases}$$

Show that $f(x)$ is discontinuous at $x = \frac{1}{2}$.

Solution: $\lim_{x\to \frac{1}{2}-} f(x) = \lim_{x\to \frac{1}{2}-} \left(\frac{1}{2} - x\right) = \frac{1}{2} - \frac{1}{2} = 0$

and $\lim_{x\to \frac{1}{2}+} f(x) = \lim_{x\to \frac{1}{2}+} \left(\frac{3}{2} - x\right) = \frac{3}{2} - \frac{1}{2} = 1$.

Since $\lim_{x\to \frac{1}{2}-} f(x) \neq \lim_{x\to \frac{1}{2}+} f(x)$, so the function $f(x)$ is discontinuous at $x = \frac{1}{2}$ and this is a discontinuity of first kind.

Here the jump of discontinuity is

$$\lim_{x\to \frac{1}{2}+} f(x) - \lim_{x\to \frac{1}{2}-} f(x) = 1 - 0 = 1.$$

ii) **Discontinuity of Second Kind:** In this case either one or both the limits do not exist.

Example: $f(x) = \begin{cases} \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Here $\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \sin\frac{1}{x}$, does not exist.

Again $\lim_{x\to 0-} f(x) = \lim_{x\to 0-} \sin\frac{1}{x}$, does not exist.

So $f(x)$ is discontinuous at $x = 0$ and this is a discontinuity of second kind.

iii) **Removable Discontinuity** :→ In this case, both the limits $\lim_{x \to c+} f(x)$ and $\lim_{x \to c-} f(x)$ exist and equal but not equal to $f(c)$.

Example :→ Consider the function $f: \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{\sin x}{x}, & \text{if } x \neq 0 \\ 5, & \text{when } x = 0 \end{cases}$$

Now $\lim_{x \to 0+} f(x) = \lim_{x \to 0+} \frac{\sin x}{x} = 1$.

and $\lim_{x \to 0-} f(x) = \lim_{x \to 0-} \frac{\sin x}{x} = 1$.

$\therefore$ $\lim_{x \to 0+} f(x) = \lim_{x \to 0-} f(x) \neq f(0)$.

$\therefore$ The function $f(x)$ is discontinuous at $x = 0$ and this is a removable discontinuity.

Here if we define $f(0) = 1$, then the function will be continuous. That's why this type of discontinuity is known as removable discontinuity.

Example :→ $f(x) = \begin{cases} x+1, & \text{when } x \leq 1 \\ 3 - ax^2, & \text{when } x > 1 \end{cases}$. For what value of 'a', will the function f be continuous at $x = 1$?

Solution :→

Here $\lim_{x \to 1-} f(x) = \lim_{x \to 1-} (x+1) = 1+1 = 2$.

and $\lim_{x \to 1+} f(x) = \lim_{x \to 1+} (3 - ax^2) = 3 - a$

Also $f(1) = 1+1 = 2$.

Now the function $f(x)$ will be continuous at $x = 1$, if $\lim_{x \to 1-} f(x) = \lim_{x \to 1+} f(x) = f(1)$.

i.e. if $3 - a = 2$

$\Rightarrow a = 1$.

$\therefore$ For $a = 1$, the given function f will be continuous at $x = 1$.

Exercises:→

1. Find the value of c, so that the function
$$f(x)=\begin{cases}\dfrac{1-\cos 4x}{8x^2}, & \text{if } x\neq 0\\ c, & \text{if } x=0\end{cases}$$
be continuous at $x=0$.

2. If the function $f(x)$ be defined by
$$f(x)=\begin{cases}3ax+b, & \text{if } x>1\\ 11, & \text{if } x=1\\ 5ax-2b, & \text{if } x<1.\end{cases}$$
be continuous at $x=1$, find a and b.

3. A function $f(x)$ is defined as
$$f(x)=\begin{cases}-2\sin x, & -\pi\leq x<-\frac{\pi}{2}\\ a\sin x+b, & -\frac{\pi}{2}\leq x<\frac{\pi}{2}\\ \cos x, & \frac{\pi}{2}\leq x<\pi.\end{cases}$$
If $f(x)$ is continuous on $[-\pi,\pi]$, then find out the values of a and b

4. Show that the function $f(x)=|x+1|+|x-2|$ $(x\in\mathbb{R})$ is continuous both at $x=-1$ and $x=2$.

5. Examine the continuity of the function
$$f(x)=\begin{cases}x\sin\frac{1}{x}, & \text{if } x\neq 0\\ 0, & \text{if } x=0\end{cases}$$ at $x=0$.

6. Using '$\delta-\varepsilon$' method, show that
$$f(x)=\begin{cases}x\cos\frac{1}{x}, & \text{if } x\neq 0\\ 0, & \text{if } x=0\end{cases}$$
is continuous at $x=0$.

7. A function f is defined as follows:
$$f(x)=\begin{cases}5x-4, & \text{for } 0<x<1\\ 4x^2-3x, & \text{for } 1\leq x<2\\ 3x+4, & \text{for } x\geq 2\end{cases}$$
Examine the continuity of f at $x=1$ and $x=2$.

8. f is defined in $(0,2)$ by $f(x)=x-[x]$. Prove that f is not continuous at $x=1$.